

# A novel model of wave turbulence

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A novel D-model of wave turbulence is presented which allows to reproduce in a single frame various nonlinear wave phenomena such as intermittency, formation and direction of energy cascades, possible growth of nonlinearity due to direct energy cascades, etc. depending on the initial state. No statistical assumptions are used, all effects are due to the behavior of distinct modes. Classical energy spectra  $E_\phi \sim \phi^{-\nu}$ ,  $\nu = \text{const} > 0$ , for dispersion function of the form  $\phi \sim k^\alpha$ ,  $\alpha > 0$  are obtained as a particular case of a more general form of energy spectra:  $E_\phi \sim \phi^{-\nu}$ ,  $2 + \alpha^{-1} \leq \nu \leq 2(2 + \alpha^{-1})$  where magnitude of  $\nu$  is defined by the parameters of the initial excitation. D-model is a generic model which can be expanded into a hierarchy of more refined models including dissipation, forcing, etc. D-model can be applied to the experimental and theoretical study of numerous wave turbulent systems appearing in hydrodynamics, nonlinear optics, electrodynamics, convection theory, etc.

## 1. Introduction.

Stationary energy spectra in the statistical theory of wave turbulence follow the celebrated Kolomogorov-Zakharov (KZ) law  $E_k \sim k^{-\nu}$ , [1], with dispersion relation  $\omega \sim k^\alpha$ ,  $\nu, \alpha > 0$  being constants, and  $k$  the wave number. These spectra are obtained under certain assumptions – like the smallness of non-linearity, infinite boxes, existence of an inertial range ( $k_1 < k < k_2$ ), local interaction in  $k$ -space etc. – in close analogy to ordinary turbulence by replacing vortices with waves.

Attempts to verify this theory led, however, to controversial results (for a review see [2]), among them in several cases the lack of the generation of the energy cascade and instead the production of regular wave patterns, [3]. Moreover, if a cascade occurs, its spectrum consists of two distinct parts: a discrete spectrum and a continuous spectrum. The former, and frequently even the continuous spectra do not follow the KZ law. Well-known examples are the elastic thin vibrating steel plate, [4], and gravity waves in mercury, [5]. Also, surface wave waves produced in laboratory in a flume of size  $12 \times 6 \times 1.5$  m developed a strongly non-linear discrete energy cascade lacking any continuous spectrum, [6]. The form of the energy spectra depends, in addition, on the parameters of the initial excitation, [7, 8]. In experiments with capillary waves in He strong non-locality was found; the local wave-amplitude maximum appeared at frequencies  $\omega$  of the order of the viscous cut-off, [9].

Attempts on improving the statistical wave turbulence theory either refer to frozen [10], sandpile [11], mesoscopic [12] or finite-dimensional models [13], neither allowing for the inclusion of the dependence on the initial conditions or finite sizes. In the present Letter we propose a novel model (called further on D-model) based on discrete wave turbulence [14, 15] and including energy cascades. It provides a uniform frame for study-

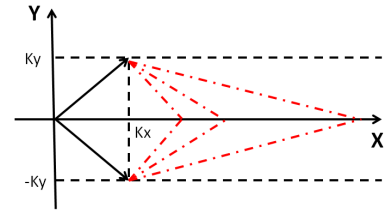


FIG. 1: Color online. Nonlocal interactions in a 4-wave system,  $\phi \sim k^\alpha$ . Each couple of (red) dashed lines with equal lengths correspond to specific choice of a parameter  $t$ .

ing finite-size effects and deducing the generic form of turbulent energy spectra depending on the parameters of initial state.

## 2. D-model.

### 2.1. Excitation of a (quasi-) resonant mode.

2.1.1. Within the discrete model of wave turbulence [16] the solutions of the resonance conditions form a group of independent resonance clusters, each cluster being graphically represented by its non-linear resonance diagram (NR-diagram) which explicitly defines the dynamical system of the cluster and its conservation laws. The solutions of the corresponding dynamical system may be either regular or chaotic, depending on the structure of the cluster and the details of the initial excitation. This holds for quasi-resonance with small frequency mismatches.

2.1.2. (Quasi-) resonant modes occur all-over  $k$ -space; they are not restricted to the range  $k < k_1$ .

2.1.3. (Quasi-) resonance interactions may not be localized in  $k$ -space. In a 4-wave system with dispersion function  $\phi \sim k^\alpha$  modes with *arbitrary large wavelength difference* can interact directly (see Fig.1). In this case a parametric series of solutions of resonance conditions  $k_1^\alpha + k_2^\alpha = k_3^\alpha + k_4^\alpha$ ,  $\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4$ , can be easily written out as  $\vec{k}_1 = (k_x, k_y)$ ,  $\vec{k}_2 = (t, -k_y)$ ,  $\vec{k}_3 = (k_x, -k_y)$ ,  $\vec{k}_4 = (t, k_y)$ , with real parameter  $t$ .

2.1.4. In addition, non-resonant modes, i.e. modes

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of large frequency mismatch, do exist. In fact, most of the modes in a 3-wave system are indeed non-resonant. These modes do not change their energies on the corresponding resonant time-scale.

## 2.2. Excitation of a non-resonant mode.

In this case the generic mechanism of instability can be described as an interaction of three monochromatic wave trains  $\phi_+ + \phi_- = 2\phi_0$  where

$$\phi_+ = \phi_0 + \Delta\phi, \quad \phi_- = \phi_0 - \Delta\phi, \quad 0 < \Delta\phi < 1. \quad (1)$$

This type of instability is quite general and is known in various areas of physics under different names, e.g. parametric instability in classical mechanics, Suhl instability of spin waves, Oraevsky-Sagdeev decay instability of plasma waves, modulation instability in nonlinear optics, Benjamin-Feir instability in deep water, etc., [16].

Conditions for the modulation instability to occur have been written in terms of the increment of instability  $I$ ; its form may differ for diverse wave systems and for the same wave system with different magnitudes of the nonlinearity parameter (e.g. [17] and [18], for surface water waves). However, as  $I$  can always be presented as a polynomial on  $\phi$ ,  $k$  and  $\Delta\phi$ , it does not affect our general scheme (presented below) for computing energy spectra. To shorten further computation we have chosen the simplest form of the instability increment according to [17] (weakly nonlinear Schrödinger equation with the value of the nonlinearity parameter  $\varepsilon \sim 0.1$  to  $0.2$ ):

$$0 < I = \frac{\Delta\phi}{\phi A k} < 1. \quad (2)$$

Aiming to construct a stationary ("saturated") energy spectrum we regard cascading chain of the form

$$\begin{cases} 2\phi_0 = \phi_{1,1} + \phi_{2,1}, & E_1 = p_1 E_0, \\ \phi_{1,1} = \phi_{2,1} + \phi_{2,2}, & E_2 = p_2 E_1, \\ \dots & \\ \phi_{n-1,1} = \phi_{n,1} + \phi_{n,2}, & E_n = p_n E_{n-1} \end{cases} \quad (3)$$

where  $0 < p_j < 1$ ,  $\phi_f$  is the forcing frequency,  $E_j$  is the energy at the  $j$ -th step of cascade and  $p_j$  is the part of the energy  $E_{j-1}$  transported from cascading mode  $A_{j-1}$  to cascading mode  $A_j$ . Each cascade step is regarded independently at discrete time moments.

**2.2.1. Assumptions.** We assume further that

(1)  $p_j = p = \text{const}$ , i.e. *cascade intensity*  $p$  is constant for given excitation parameters; in this case  $p = (1 + \sqrt{1 - E_0})/2$ , where the energy of the initial excitation is normalized to 1 and  $E_0$  is the energy of its stationary amplitude  $A_0$ .

(2) at each cascade step  $j$ , one cascading mode with frequency  $\phi_{j,1}$  is generated, with maximal increment of instability

$$I_{\text{max},n} = \frac{|(\Delta\phi)_n|}{\phi_n A_n k_n} = 1. \quad (4)$$

(this corresponds to the idea of Phillips who suggested that in the saturated range the spectral density is saturated at a level determined by wave breaking). Modes

$\phi_{j,2}$  have smaller increments of instability and are fluxless ("frozen", [10]).

**2.2.2. The chain equation.** Cascading chain consists of modes with frequencies  $\phi_{1,1}, \phi_{2,1}, \phi_{3,1}, \dots$ ; it can be shown that the distance between two cascading modes  $|(\Delta\phi)_n|$  decreases slowly with growing  $n$ . It follows from (1) that

$$E_n = p^n E_0 \Rightarrow A_{n+1} = \sqrt{p} A_n, \quad (5)$$

The second assumption (2) can be rewritten as

$$\phi_{n+1} = \phi_n \pm \phi_n A_n k_n, \quad (6)$$

and combination of  $A_{n+1} = \sqrt{p} A_n$  and (6) yields

$$\sqrt{p} A_n = A(\phi_n \pm \phi_n A_n k_n) \quad (7)$$

which is further on called *the chain equation* ("+" and "-" stand for direct and inverse cascade, accordingly).

**2.2.3. Energy spectra.** Taking Taylor expansion of RHS of the chain equation yields

$$\begin{aligned} \sqrt{p} A_n &= A(\phi_n \pm \phi_n A_n k_n) = \sum_{s=0}^{\infty} \frac{A_n^{(s)}}{s!} (\pm \phi_n A_n k_n)^s \\ &= A_n \pm A_n' \phi_n A_n k_n + \frac{1}{2} A_n'' (\pm \phi_n A_n k_n)^2 + \dots \end{aligned} \quad (8)$$

Taking two first RHS terms from (8) we get

$$\sqrt{p} A_n = A_n \pm A_n' \phi_n A_n k_n \Rightarrow A_n' = \pm \frac{\sqrt{p} - 1}{\phi_n k_n} \Rightarrow \quad (9)$$

$$A(\phi) = \pm (\sqrt{p} - 1) \int \frac{d\phi}{\phi k} + \text{const}(\phi_0, A_0) \quad (10)$$

Substitution of some specific dispersion relation into (9) yields dependence  $A = A(\phi)$  and the form of energy spectrum,  $E \sim A^2$ . E.g. if  $\phi = k^\alpha$ ,  $\alpha > 0$ , (10) yields

$$\begin{aligned} A(\phi) &= \mp \frac{(1 - \sqrt{p})}{2 + \alpha^{-1}} (\phi^{-(2+\alpha^{-1})} - \phi_0^{-(2+\alpha^{-1})}) + A_0, \\ E(\phi) &\sim \phi^{-\nu} \quad \text{with} \quad 2 + \alpha^{-1} \leq \nu \leq 2(2 + \alpha^{-1}). \end{aligned}$$

This is in accordance with the laboratory results reported e.g. in [6, 7], for  $\phi \sim k^{1/2}$ .

**2.2.4. Direction and termination of cascades.** Conditions for cascade's termination and for formation of the direct and inverse cascades can be easily obtained from the form of the increment (6). Namely, cascade terminates at some step  $N$  if (2) is violated:

$$\phi_{N+1} - \phi_N = 0 \quad \text{or} \quad |\phi_{N+1} - \phi_N| > \phi_N A_N k_N. \quad (11)$$

Direct and inverse cascades occur if

$$\phi_{n+1} - \phi_n > 0 \quad \text{or} \quad \phi_n - \phi_{n+1} > 0 \quad (12)$$

accordingly. Depending on the value of excitation parameters,  $N < \infty$  or  $N = \infty$ , [19].

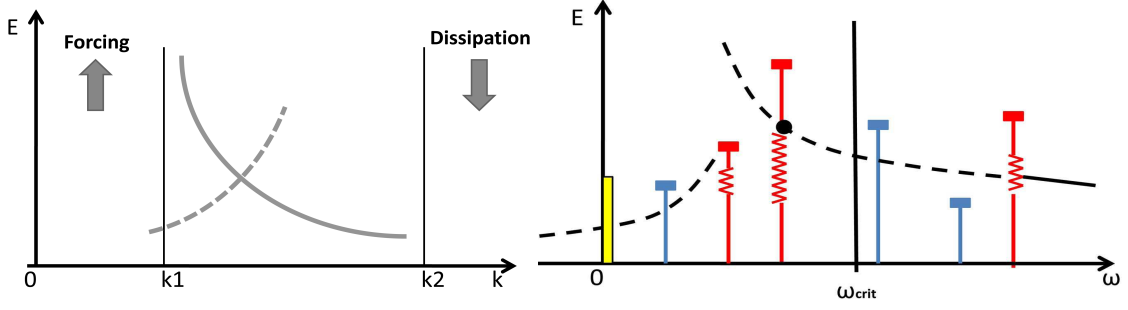


FIG. 2: Color online. Classical model of WT (on the left) and D-model of WT (on the right).

A simplified form of cascade with  $|(\Delta\phi)_n| = \text{const}$ , [20], though useful for interpreting some experimental results, yields cascade's termination at the final step and does not give a lead for possible transition to continuous spectrum.

### 3. Classical model *versus* D-model.

Graphical presentation of the classical model and D-model of WT is shown in Fig.2 (in the left and right panels correspondingly). One of crucial assumptions of the classical model is, that forcing and dissipation are wide apart in the  $k$ -space, while the energy transport described by stationary spectra takes place within inertial interval  $(k_1, k_2)$ . This assumption is very difficult to reproduce either in laboratory experiments or numerical simulations. "Finite-size" effects are supposed to appear in the range of wavelengths  $k < k_1$  and are not described by the classical model, while in the range  $k > k_2$  energy transport is supposed to be suppressed by dissipation. This phenomenology does not depend on the details of the initial forcing.

On the other hand, in the D-model dissipation might be switched on or off at any cascade step by changing the magnitude of cascade intensity  $p$ , i.e. no additional assumptions are needed about relative positions of forcing and dissipation in the  $k$ -space; the existence of inertial interval is not important either and initial excitation may take place all over the  $k$ -space. In this model, three types of modes are singled out; energy transport in wave system depends on the type of the initially excited mode.

**3.1. Resonant and quasi-resonant modes** (shown as red vertical solid T-shaped lines having a spring-like part); being excited, they may yield chaotic or recurrent patterns; no cascade shall be generated (e.g. [3]). In this case *time synchronization* condition (for frequencies) is satisfied exactly or with a small frequency mismatch while *space synchronization* condition (for wave vectors) is satisfied exactly:

$$\begin{aligned} \phi_1 + \phi_2 &= \phi_3 + \Omega, \quad \vec{k}_1 + \vec{k}_2 = \vec{k}_3, \quad 0 \leq \Omega \ll 1 \text{ or} \\ \phi_1 + \phi_2 &= \phi_3 + \phi_4 + \Omega, \quad \vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4, \end{aligned}$$

and phases are coherent.

**3.2. Cascading modes** (shown as small dashes composing black dashed curves); being excited, they form a cascading chain with energy discrete spectrum  $\sim A^2$  with  $A$  computed from (10); for some specific initial magnitudes  $A_0$  and  $\phi_0$  energy spectrum might turn into continuous (shown as black solid "tail" of the curve) though not necessarily a classical spectrum. Choice of the parameters of initial excitation also defines whether excitation of a cascading mode yields direct or inverse cascade (e.g. [22]). In this case *the time synchronization* condition (1) is satisfied exactly, while *the space synchronization* condition is violated:

$$\phi_1 + \phi_2 = 2\phi_3, \quad \vec{k}_1 + \vec{k}_2 = 2\vec{k}_3 + \Theta, \quad 0 \leq \Theta \ll 1,$$

and phases are not coherent.

**3.3. Frozen modes** (shown as blue vertical solid T-shaped lines): these are fluxless modes which do not take part in energy transport over the  $k$ -space, they may occur for some choice of initial state (e.g. [10, 21]). For frozen modes *both time and space synchronization conditions are violated*. This means in particular that fluxless modes  $\phi_{1,2}, \phi_{2,2}, \phi_{3,2}, \dots$  have non-coherent phases and provide a necessary prerequisite in the statistical theory for a possible further development of a classical spectra. If at some cascade step a (quasi-) resonant mode is generated (shown as a black circle, Fig.2, on the left) a cascading chain may be broken by appearance of an intermittency.

It is important to realize that increment of instability is a parameter characterizing interplay of dispersion and nonlinearity in a wave turbulent system. Accordingly, two conditions of cascade termination (11) have different physical consequences:  $\phi_{N+1} - \phi_N = 0$  yields  $I = 0$  and manifests transition to a linear regime, while  $|\phi_{N+1} - \phi_N| > \phi_N A_N k_N$  corresponds to the case  $I > 1$  and growing nonlinearity. For instance, it has been shown in [19] for surface water waves that starting with a small nonlinearity  $\varepsilon_0 = A_0 \phi_0 = 0.1$ , initial amplitude loses an order in the magnitude after 31 steps of the cascading chains while the steepness of a wave packet becomes 0.288. This means that another form of the increment should be used further on, e.g.

$$I = |\Delta\phi| / \left( \phi A k - \frac{3}{2} \phi^2 A^2 k \right), \quad (13)$$

obtained in [18] for weakly nonlinear Schrödinger equation with parameter of nonlinearity  $\varepsilon \sim 0.25$  to  $0.4$ . This means also that the condition  $|\phi_{N+1} - \phi_N| > \phi_N A_N k_N$  allows to compute a critical frequency  $\phi_{crit}$  (shown as a (black) bold vertical line at Fig., on the right) at which nonlinearity becomes mode important than dispersion. The energy spectra  $E(\phi)$ ,  $\phi > \phi_{crit}$  would differ substantially from the classical spectra.

Another interesting phenomenon observed in numerous laboratory experiments is formation of a narrow zero-frequency sideband with non-zero energy (shown as vertical yellow rectangular). It should be checked whether this phenomenon might be attributed to termination of an inverse cascade or originates in the (quasi) resonant interactions of short eigen-modes (appearance of an intermittency).

#### 4. Conclusions.

In this Letter our main goal was to understand natural phenomena revealed by laboratory observations of WT and to find the physical mechanism that underlies it. We have developed a simple model and analyzed it to see if it captures critical aspects of the phenomenon.

We have demonstrated that D-model of WT allows to reproduce a number of discrete effects including generation of a set of cascading modes in a wave turbulent system. We have also deduced the corresponding energy spectrum depending on the form of dispersion relation  $\phi = \phi(k)$ , the form of the instability increment and parameters of the initial excitation. Conditions for formation of a direct or an inverse cascade have been written out explicitly. It has been shown that, depending on ini-

tial state, a cascade might yield decrease or growth of the nonlinearity.

A few possible ramifications of the model are accounted for below (the list is not exhaustive):

- to change the form of the instability increment (2), e.g. to (13); a number of examples is given in [16];
- to regard cascade intensity as a function of wavelengths,  $p = p(k) \neq \text{const}$ , then ODE similar to (9) is also solvable in quadratures:

$$A(\phi) = \pm \int \frac{(\sqrt{p(k)} - 1)d\phi}{\phi k} + \text{const}(\phi_0, A_0). \quad (14)$$

For instance, power function  $p(k_j) = k_j^\gamma$  may model dissipation; piecewise constant function  $p = p_j, \forall k \in (k_{j-1}, k_j)$  may model wind-generated surface water waves at different fetches  $(k_{j-1}, k_j)$ , etc.

- to regard the hierarchy of D-models with refined form of turbulent energy spectra obtained by cutting the Taylor expansion (8) at 3, 4 and so on terms instead of the simplest equation (9). It would be a challenge to find the general solution of the infinite order ODE given by (8) and to investigate bounds of its physical applicability (it should be verified that energy flux corresponding to chosen ODE is not linear).

**Acknowledgements.** Author acknowledges M. Shats, I. Shugan and R. Treumann for valuable discussions. This research has been supported by the Austrian Science Foundation (FWF) under project P22943-N18 "Nonlinear resonances of water waves" and in part by the National Science Foundation, USA, under Grant No. NSF PHYS05-51164.

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